Abel-Jacobi map.

X smooth proj $/\mathbb{C}$ $\quad J^r(X) = H^{2r-1}(X, \mathbb{C}) / F^r H^{2r-1} + H^{2r-1}(X, \mathbb{Z})$.

$A^r =$ codim r-cycle alg. equiv. to 0.

$\Theta : A^r(X) \longrightarrow J^r(X)$.

Thm: Let X be a smooth quantic 3-fold. Then $\Theta : A^2(X) \to J(X)$ is an isomorphism.

We say Θ is an isogeny if Θ is surjective w/ finite kernal.

Lemma: $\pi : Y \to X$ blow-up along a sm. subvariety. $\dim X = \dim Y = 3$.

Θ_X is an isogeny $\Longleftrightarrow$ Θ_Y is an isogeny.

pf:

Lemma: $\pi : Y \longrightarrow X$ gen. finite deg d.

Θ_Y isogeneny $\Longrightarrow$ Θ_X is an isogeny.

pf:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \Theta_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & A^2(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & J(X) \arrow[r] \arrow[d, "\pi^*"] & {} \\
0 \arrow[r] & \operatorname{Ker} \Theta_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & A^2(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & J(Y) \arrow[r] \arrow[d, "\pi_*"] & {} \\
0 \arrow[r] & \operatorname{Ker} \Theta_X \arrow[r] & A^2(X) \arrow[r] & J(X) \arrow[r] & 0
\end{tikzcd}$$

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$\pi_* \pi^* = d \cdot Id$. So $\pi_* \pi^*$ is surj

for $J(X) \to J(X)$. $\Rightarrow J^2(Y) \xrightarrow{\pi_*} J^2(X)$ surj

$$\begin{tikzcd} A^2(Y) \arrow[r, two heads] \arrow[dr] & J^2(Y) \arrow[r] & J^2(X) \\ & A^2(X) \arrow[ur] & \end{tikzcd}$$ is surj.

$\Rightarrow A^2(X) \twoheadrightarrow J^2(X)$.

$A^2(X)$ & $A^2(Y)$ is divisible $\Rightarrow$ $A^2(X) \to A^2(Y) \to A^2(X)$.
$\underbrace{\qquad\qquad}_{\text{surj.}}$

$$\begin{tikzcd} & & 0 & \\ \mathrm{Ker}\,\Theta_{\cancel{X}} \arrow[r] \arrow[d] & \mathrm{Ker}\,\Theta_X \arrow[r] & K \arrow[r] \arrow[u] & 0 \\ \mathrm{Ker}(\Theta_Y) \arrow[ur] & & \mathrm{Ker}(J(X) \to J(X)) \arrow[u] & \end{tikzcd}$$

Image of $\mathrm{Ker}\,\Theta_X \to \mathrm{Ker}\,\Theta_X$ is finite!

Lemma: $\pi: V \to W$ conic bundle. i.e.
$\exists\, W^0 \subseteq W$ open s.t $\pi: \pi^{-1}(W^0) \to W^0$
~~is a sm morphism~~ $\pi^{-1}(W) \hookrightarrow \mathbb{P}^2 \times W^0$
$\searrow W^0 \swarrow$

family of conics.

If $A_0(W) \to Alb(W)$ is an isogeny.
Then $A^2(V) \xrightarrow{\Theta_V} J^2(V)$ is an isogeny.

pf: $\exists\ W'$ s.t

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\phi"] & \\ W' \times \mathbb{P}^1 \arrow[rr, dashed] \arrow[d, "p"'] & & U \arrow[d, "\pi"] \\ W' \arrow[rr, "f"] & & W \end{tikzcd}$$

Rmk: $A_0(W') \to Alb(W')$ has huge kernal!

$T(W') = \mathrm{Ker}(A_0(W') \to Alb(W'))$ divisible.

$$\begin{tikzcd} T(W') \arrow[r] \arrow[d] & T(W) \arrow[d] \\ A_0(W') \arrow[r] \arrow[d] & A_0(W) \arrow[d] \\ Alb(W') \arrow[r] \arrow[d] & Alb(W) \arrow[d] \\ 0 & 0 \end{tikzcd}$$

$f_* T(W') \subset T(W)$ is finite but also divisible!

$\Rightarrow f_* T(W') = 0$.

$T(W') \simeq \mathrm{Ker}(A^2(W' \times \mathbb{P}^1) \to J^2(W' \times \mathbb{P}^1))$.

$\simeq \mathrm{Ker}(A^2(X) \to J^2(X))$.

$$\begin{tikzcd} 0 \arrow[r] & K(U) \arrow[r] \arrow[d] & A^2(U) \arrow[r] \arrow[d] & J^2(U) \arrow[d] & \\ 0 \arrow[r] & T(W') \arrow[r] \arrow[d] & A^2(X) \arrow[r] \arrow[d] & J^2(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K(U) \arrow[r] & A^2(U) \arrow[r] & J^2(W) \arrow[r] & 0 \end{tikzcd}$$

b/c $A^2(W' \times \mathbb{P}^1) \to J^2(W' \times \mathbb{P}^1)$ is surj.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & W' \times_W U \arrow[r] \arrow[d] & U \arrow[d] \\ & W' \arrow[r, "f"] & W \end{tikzcd}$$

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Geometric construction.

$x = [1,0,0,0,0] \in \mathbb{P}^4$ $x \in X$.

$$X_0^3 L(X_1, \dots, X_4) + X_0^2 Q(X_1, \dots, X_4) + X_0 C(\;) + \cancel{G_4} = 0.$$

Assume $L = Q = 0$ defines a sm conic in $\mathbb{P}^2$ (true if $x \in X$ is general).

~~$y \in \mathbb{P}^3(y_1, y_2, y_3, y_4)$ s.t~~

$y = [0, y_1, y_2, y_3, y_4]$ s.t $L(y) = Q(y) = 0$.

Line $\overline{xy} \cap X = 3 \cdot \{x\} + \{z\}$.

$\Rightarrow$ conic bundle over $\begin{matrix}\mathcal{C}\\ \downarrow\\ X\end{matrix} \longrightarrow X$.

$(x, y) \longmapsto z$.

$\exists$ a ruled surface $\Sigma \subset X$ s.t $\begin{matrix}\mathcal{C}|_\Sigma\\ \downarrow\\ \Sigma\end{matrix} \to X$ is dominat.

$V \supset X$ a general quartic 4-fold in $\mathbb{P}^5$

$X = V \cap H \subset V \subset \mathbb{P}^5$.

S = Fano variety of lines of V.

S is a sm ~~surface~~ 3-fold b/c V is general.

$$\begin{tikzcd}
 & X \arrow[r, hook, "i"] & V \\
L(C) \arrow[ur] \arrow[r, hook] \arrow[d] & P \arrow[u, "p_1"] \arrow[r, hook, "j"] \arrow[d, "p_2"] \arrow[ur, phantom, "\square"] & L(S) \arrow[u, "q_1"] \arrow[dl, "q_2"] \\
C \arrow[r] & S &
\end{tikzcd}$$

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$P \to S$ is birational. (若 C is smooth. $P = Bl_C S$.

(*) $p_{1*}: H^3(P, \mathbb{Z}) \to H^3(X, \mathbb{Z})$ is surj.

若 P is singular, apply to a resolution or $H_3(P, \mathbb{Z})$?

(**) $p_{1*} p_2^* H^3(S, \mathbb{Z}) = 0$ in $H^3(X, \mathbb{Z})$.

$$p_{1*} p_2^* H^3(S, \mathbb{Z}) = p_{1*}\left(H^3(S, \mathbb{Z}) \xrightarrow{q^*} H^3(L(S), \mathbb{Z}) \xrightarrow{j^*} H^3(P, \mathbb{Z})\right)$$

$$= p_{1*} j^* q^* H^3(S, \mathbb{Z})$$

$$= i^* q_{1*} q_2^* H^3(S, \mathbb{Z}) = 0.$$

Lemma: Let V be a smooth. proj. variety $/\mathbb{C}$. $\dim V = d+1$.

$\phi: W \to V$ proper morphism. gen. finite. W irr.

$X \hookrightarrow V$ sm. hyperplane section.

$Y = \phi^{-1}(X)$. then

$\operatorname{Im} \phi_*: H_d(Y, \mathbb{Z}) \to H_d(X, \mathbb{Z})$

contains $\operatorname{Ker}(H_d(X, \mathbb{Z}) \to H_d(V, \mathbb{Z}))$.